Optimal Global Conformal Surface Parameterization

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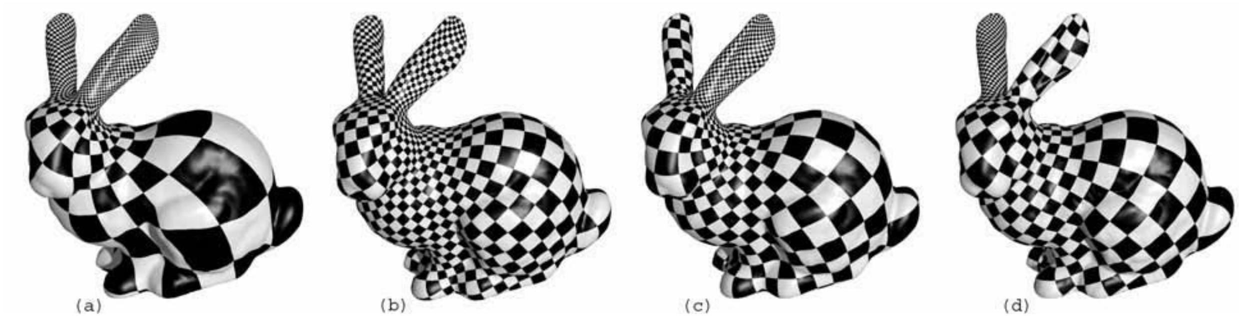

Figure 1: Uniform global conformal parameterization ((a) and (b)) and region emphasized conformal parameterization ((c) and (d)). (a). Least uniform conformal parameterization with energy: $21.208e-5$. (b). Most uniform conformal parameterization with energy: $3.685e-5$. (c). Maximizing the parameter area of the left half surface (with percentage: 83.48%). (d). Maximizing the parameter area of the right half surface (with percentage: 82.58%.)

Abstract

All orientable metric surfaces are Riemann surfaces and admit global conformal parameterizations. Riemann surface structure is a fundamental structure and governs many natural physical phenomena, such as heat diffusion and electro-magnetic fields on the surface. A good parameterization is crucial for simulation and visualization. This paper provides an explicit method for finding optimal global conformal parameterizations of arbitrary surfaces. It relies on certain holomorphic differential forms and conformal mappings from differential geometry and Riemann surface theories. Algorithms are developed to modify topology, locate zero points, and determine cohomology types of differential forms. The implementation is based on a finite dimensional optimization method. The optimal parameterization is intrinsic to the geometry, preserves angular structure, and can play an important role in various applications including texture mapping, remeshing, morphing and simulation. The method is demonstrated by visualizing the Riemann surface structure of real surfaces represented as triangle meshes.

CR Categories: I.3.5 [Computational Geometry and Object Modeling]: Curve, surface, solid, and object representations—Surface Parameterization

Keywords: Computational geometry and object modeling; Curve, surface, solid, and object representations; Surface parameterization.

1 Introduction

Surface parameterization is the process of mapping a surface to a planar domain and has many applications in various fields of science and engineering, including texture mapping, geometric morphing, surface matching, surface remeshing, and surface extrapolation. For example, texture mapping can be used to enhance the visual quality and generate different visual results. Geometric morphing can be used to generate vivid animation results. Essentially, surface parametrization can convert a 3D geometric problem to 2D, thereby improving the efficiency and simplifying the computation.

Conformal surface parameterizations have many merits: they preserve angular structure, are intrinsic to geometry, and are stable with respect to different triangulations and small deformations. It has been widely used for many applications, such as non-distorted texture mapping [23], [16],[20], surface remeshing [1], surface fairing [22], surface matching [14], brain mapping [2], [13] etc.

It is desirable to parameterize surfaces globally without any seams. The existence of global conformal parameterization is a non-trivial fact. This is equivalent to the fact that all orientable surfaces are Riemann surfaces. The atlas formed by the global conformal parameterization is the so-called *conformal structure*. Conformal structure is a fundamental structure between metric structure and topological structure and governs many natural physical phenomena. The abstract concept of a Riemann surface can also be visualized by texture mapping special patterns using global conformal parameterizations. This is the only means of visually conveying conformal information of surfaces.

The early work of global conformal parameterization has been done in [14, 15], where the basis for all possible global conformal parameterizations are computed. Because global conformal parameterization is non-unique, the problem of finding the optimal one remains open.

This paper introduces an explicit method to find the optimal global conformal parameterizations of arbitrary surfaces. First, the metrics for measuring the quality of conformal parameterizations are designed. Second, the major factors affecting the quality of the parameterization are summarized. Then, algorithms are developed to modify the topology, locate the zero points, and determine the cohomology types of the differential forms. The method is based

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on finite dimensional optimization and demonstrated by visualizing the Riemann surface structure of real surfaces.

1.1 Contributions

This paper introduces algorithms to optimize global conformal parameterizations. The method is based on Riemann surface theories and differential geometry. Therefore, it is rigorous and general. The optimization algorithms can be generalized to all parameterization methods based on convex combinations [10]. Our main contributions are listed as follows.

1 We introduce energy functionals on the space of complex-valued holomorphic mappings.

2 In [10], the author raised the following open question: "Under what boundary condition is a harmonic map between two topological disks conformal?" We answer this question in an algorithmic way. We compute the double covering of a topological disk (double covering means gluing two copies of the same surface along their boundaries to form a closed symmetric surface; details are described in [15]), and conformally map the double covering to a sphere preserving the symmetry. Thus the disk itself is mapped to a hemisphere. Then a conformal map between two disks is induced by their mappings to the same hemisphere. The boundary condition which makes a harmonic map conformal can be computed using this algorithm directly.

3 The difference between the zero points of a conformal parameterization and singularities of general vector fields is that the zero points cannot be arbitrarily assigned and are determined by the conformal structure. To the best of our knowledge, this statement has never been addressed in computer graphics, although it is the major topological obstruction for any surface parameterization method.

4 We propose a way of finding the optimal Möbius transform that best balances area deformations (note that conformality is invariant through Möbius transforms.)

5 This paper explains the following fact: the area stretching factor increases exponentially at the tip of long tubes and it is true for all other parameterization methods. This shows the limits of current parameterization techniques and justifies topological modification techniques proposed in this paper. Although some researchers reached the same conclusion by heuristic methods, a rigorous proof is given in this paper.

1.2 Related Work

Surface conformal parameterization algorithms have been thoroughly studied in the literature. We summarize them according to the topologies of surfaces that they can handle.

Conformal map for topological disks Many researchers propose methods to build a conformal map for topological disks. Pinkall and Polthier derive the *discrete Dirichlet energy* in [25]. Eck et al. [8] introduce the *discrete harmonic map*, which approximates the continuous harmonic maps by minimizing a *metric dispersion* criterion. Duchamp formulates the *hierarchical harmonic embedding* in [7]. Floater introduces a *shape-preserving* method in [9], which is very similar to harmonic maps for planar surfaces. Sheffer and de Sturler introduce *angle based flattening* to compute conformal maps. Desbrun et al. [1, 6] compute the discrete Dirichlet energy and apply conformal parameterization to interactive geometry remeshing. Levy et al.[23] compute a quasi-conformal parameterization by approximating the Cauchy-Riemann equation using the least square method. The above two formulations are equivalent. Hormann and Greiner propose the MIPS parameterization [18], which roughly attempts to preserve the ratio of singular values over the parameterization. Degener et al. [5] extend the method in [18] and provide a control parameter that allows for mediation between angle and area distortion.

Conformal map for genus zero closed surfaces Haker et al. [16] introduce a method to compute a global conformal mapping from a genus zero surface to a sphere by representing the Laplace-Beltrami operator as a linear system. Gu et al.[14] introduce a nonlinear optimization method to compute global conformal parameterizations for genus zero surfaces. The optimization is carried out in the tangential spaces of a sphere.

Conformal map for high genus surfaces Few researchers report their work on surfaces with complicated topology. Gu and Yau introduce algorithms to compute conformal structures determined by the metric for general closed surfaces in [14]. The proposed method approximates De Rham cohomology by simplicial cohomology and computes a basis of holomorphic 1-forms. Later the method is generalized for surfaces with boundaries in [15].

In [27, 24, 28], the Riemann surface structure is defined for combinatorial meshes. Because the metric information is ignored in their work, their methods cannot be applied to our problems directly.

2 Sketch of Mathematical Theories and Algorithm Overview

This section introduces the basic concepts in Riemann surface theory related to global conformal parameterization and an overview of the optimization algorithms.

2.1 Theoretic Background

The basic concepts of Riemann surface theories are briefly sketched. Further details can be found in [19], [12] and [26].

Conformal Chart Let U be an open set of $S \in \mathbb{R}^3$. A parameterization of U is a one to one map $z : U \rightarrow \mathbb{R}^2$, which maps U to the (u,v) plane. (U,z) is called a chart of S. In the case of conformal chart, the first fundamental form satisfies: $ds^2 = \lambda(u,v)^2(du^2+dv^2)$, where $\lambda(u,v)$ is called the *stretch factor*, a function that scales the metric at each point (u,v). The coordinate pair (u,v) is called a *conformal parameter* of the surface patch U. (U,z) is called a *conformal chart* of S.

Conformal Atlas All oriented metric surfaces are Riemann surfaces and have a global conformal atlas, or a set of conformal charts. In the following discussion, we treat $\mathbb{R}^2$ as a complex plane, where the point (u,v) is equivalent to $z = u + iv$, and $(u,-v)$ is equivalent to $\bar{z} = u - iv$. In later sections, we use both representations interchangeably.

Let S be a surface in $\mathbb{R}^3$ with an atlas $\{(U_\alpha, z_\alpha)\}$, where (U_α, z_α) is a chart, and $z_\alpha : U_\alpha \rightarrow \mathbb{C}$ maps an open set $U_\alpha \subset S$ to the complex plane $\mathbb{C}$.

The atlas is called *conformal* if **(1).** each chart (U_α, z_α) is a conformal chart. Namely, on each chart, the first fundamental form can be formulated as $ds^2 = \lambda(z_\alpha)^2 dz_\alpha d\bar{z_\alpha}$, **(2).** the transition maps $z_\beta \circ z_\alpha^{-1} : z_\alpha(U_\alpha \cap U_\beta) \rightarrow z_\beta(U_\alpha \cap U_\beta)$ are holomorphic.

A chart is compatible with a given conformal atlas if adding it to the atlas again yields a conformal atlas. A *conformal structure* (*Riemann surface structure*) is obtained by adding all compatible charts to a conformal atlas. A *Riemann surface* is a surface with a conformal structure.

Holomorphic 1-form Given a Riemann surface S with a conformal atlas $\{(U_\alpha, z_\alpha)\}$, a holomorphic 1-form ω is defined by a family $\{(U_\alpha, z_\alpha, \omega_\alpha)\}$, such that **(1)** $\omega_\alpha = f_\alpha(z_\alpha)dz_\alpha$, where f_α is holomorphic on U_α, and **(2)** if $z_\alpha = \phi_{\alpha\beta}(z_\beta)$ is the coordinate transformation on $U_\alpha \cap U_\beta (\neq \emptyset)$, then $f_\alpha(z_\alpha)\frac{dz_\alpha}{dz_\beta} = f_\beta(z_\beta)$, the local representation of the differential form ω satisfies the chain rule.

For a Riemann surface S with genus $g > 0$, all holomorphic 1-forms on S form a complex g-dimensional vector space ($2g$ real dimension), denoted as $\Omega^1(S)$. The quality of a global conformal parameterization for a high genus surface is mainly determined by the choice of the holomorphic 1-form.

The *zero points* of a holomorphic 1-form ω are the points where, on any local representation $(U_\alpha, z_\alpha, \omega_\alpha)$, ω_α equals zero. For a genus $g > 0$ surface, there are in general $2g-2$ zero points for each holomorphic 1-form.

Möbius Transformation Group For genus zero closed surfaces, there is no holomorphic one form. The global conformal parameterization is a conformal map $\phi : S \to S^2$ from the surface S to the unit sphere S^2. Two such kinds of transformations differ by a Möbius transformation on S^2. Suppose both ϕ_1 and ϕ_2 are two conformal parameterizations of S, consequently $\phi_2 \circ \phi_1^{-1} = \mu$, where μ is a Möbius automorphism of the sphere. All conformal maps from S to S^2 can then be formulated as $\mu \circ \phi_1$. We compute one conformal map ϕ_1 first, then compose it with a Möbius transformation μ. By choosing an appropriate μ, we can optimize the energy.

A genus zero open surface can be globally conformally parameterized by the unit disk. Two such kinds of parameterizations differ by a Möbius transformation defined on the disk. We can find the best one with a similar method to that used for a genus zero closed surface.

2.2 Optimization Algorithms Overview

In order to measure the quality of a global conformal parameterization, we define different metrics for different applications. There are several main factors affecting the quality of a parameterization, including the topology of the surface, the zero point position and the choice of the holomorphic 1-form for a high genus surface or the Möbius transformation for a genus zero surface. The algorithms for optimizing these factors are as follows.

- *Metric for parameterization*. We formulate different functionals to measure the qualities of parameterizations, including uniformity energy, parameter area of emphasized regions and zero points locations.
- *Topology Optimization*. The long tube shape causes an exponential shrinking parameterization. We design a method to mediate this problem.
- *Zero Point Allocation*. The parameterization near the zero points is singular; it is desirable to allocate zero points at the predefined positions.
- *Optimal Holomorphic 1-form*. The global conformal parameterization for a higher genus surface is induced by a holomorphic 1-form. The special holomorphic 1-form is chosen to optimize the functional for parameterizations.
- *Optimal Möbius Transform*. The global conformal parameterization of a genus zero surface is determined by a Möbius transform. The algorithm is designed to find an optimal Möbius transform to maximize the functional.

2.3 Approximation Strategy

The concepts of Riemann surfaces are defined for smooth surfaces. In practice, it is impossible to represent the smooth surface and conformal structure using finite memory. We approximate them by the finite element method. Specifically, we approximate a smooth surface S by a series of piecewise linear triangular meshes $\{\tilde{S}_n\}$ such that each $\tilde{S}_n$ doesn't have many obtuse angles and approximate the smooth conformal structures of S using piecewise linear mappings defined on $\{\tilde{S}_n\}$. Such discrete mappings are called *discrete holomorphic 1-forms*. The existence of $\{\tilde{S}_n\}$ has been shown in [15].

A natural question arises of whether the approximation converges to the real conformal structure of the smooth surface. The answer is positive. Computing conformal structure is equivalent to solving an elliptic Partial Differential Equation (PDE) on the surface. It has been proven in finite element field that the discrete approximation converges to the real solution [3]. Also, the solutions to elliptic PDEs are stable and smooth in general. This implies the convergence and stability of our approximation.

Because of the convergence and the stability of our discrete approximations, they behave like the real solutions asymptotically. In the following discussion, we conceptually treat them as smooth solutions and do not differentiate discrete approximation and smooth solution.

There is another important point we want to clarify. The conformal structure is determined by the metric of the surface. Even if a surface is not smooth, such as the mesh $\tilde{S}_n$ in the approximation, it still has a smooth metric and a smooth conformal structure. The discrete holomorphic 1-forms in the approximations are not the real conformal structure of the mesh $\tilde{S}_n$.

3 Algorithms for Global Conformal Parameterization Optimization

In our current work, the surfaces are represented as meshes. Suppose K is a simplicial complex, and a mapping $r : |K| \to R^3$ embeds $|K|$ in R^3. $M = (K, r)$ is called a *triangular mesh*. K_n, where $n = 0, 1, 2$, are the sets of *n-simplices*. We use $[v_0, v_1, \cdots, v_n]$ to denote a n-simplex, where $v_i \in K_0$.

We use these symbols in the following discussion: E - energy for a parameterization, ω - a holomorphic 1-form, λ_i - the coefficients of ω, λ - conformal factor, τ - the stereo-graphic projection, μ - a Möbius transformation, ϕ - a conformal map between surfaces.

3.1 Computing Conformal Structures

We use the methods introduced in [14], [15] to compute conformal structures.

Genus 0 closed surfaces can be conformally parameterized over a unit sphere, and harmonic maps of these surfaces are equivalent to conformal maps. We use a Gauss map as the initial map, and then we use the heat flow method to reduce the harmonic energy with special constraints. The final harmonic map is a global conformal parameterization. By composing it with a Möbius map of the sphere, we can obtain all possible global conformal parameterizations.

For genus 0 open surfaces, we use double covering to get a closed symmetric surface. We can map this double covered surface conformally to a sphere and preserve the symmetry; i.e, each copy of the original surface is mapped to a hemisphere. Then we use stereographic projection to map a hemisphere to a unit disk; the surface is globally conformally parameterized by the disk. By composing with a Möbius map of the disk, we can construct all global conformal parameterizations for the surface.

The conformal structure of a higher genus surface can be represented as a holomorphic one-form basis, which is a set of $2g$ functions $\omega_i : K_1 \to R^2, i = 1, 2 \cdots, 2g$. Any holomorphic one-form ω is a linear combination of these functions. The surface can be cut open to a topological disk, namely a *fundamental domain*. Verdiére et al. [4] and Lazarus et al. [21] discussed the algorithms for computing fundamental domains for general surfaces. By integrating ω on a fundamental domain, the whole surface can be globally conformally mapped to the *uv* plane.

The computation process for $\{\omega_i, i = 1, 2, \cdots, 2g\}$ can be summarized as computing the homology basis, cohomology basis, harmonic one-form basis and holomorphic one-form basis.

Double covering techniques are applied to surfaces with boundaries to convert them to closed symmetric surfaces. Therefore, in the following discussion, we assume the surfaces are closed.

3.2 Metrics for parameterization

In order to convert the whole mesh to a geometry image or spline surface patches, parameterizations with high uniformity are preferred. It is often desirable to allocate more parameter areas for special regions on the surface in real applications. For example, in surface remeshing, more samples are required for regions with high Gaussian curvature or sharp features. Sometimes, multi-chart geometry images are used to represent the shape. In this case, we can use several global parameterizations, each of which will emphasize a surface region and convert it to one chart in the geometry image. Also in this scenario, the parameterization emphasizing different regions are also desirable. For high genus surfaces, the existence of zero points is unavoidable, and the neighborhoods of zero points will be under sampled in the parameter domain. Therefore, users would like to assign the zero points to positions that have lower curvature or are less visible. In order to allocate zero points at the prescribed positions, we design a special metric to measure the parameter area of the neighborhoods of the given points. If the parameterization of zero points places them at the desired positions, this metric will be close to zero.

Suppose $\Omega \subset R^2$ is the parameter domain for a surface S and (u, v) are parameters on Ω. Then the functional for measuring uniformity is

$$E = \int_{\Omega} (\lambda(u,v)^2 - 1)^2 du dv, \tag{1}$$

where λ is the conformal factor, subject to

$$\int_{\Omega} du dv = \int_{\Omega} \lambda(u,v)^2 du dv. \tag{2}$$

Similarly, suppose Ω is divided into two regions Ω_1 and Ω_2, and we would like to emphasize Ω_1. Then the functional is

$$E = \int_{\Omega_1} \lambda(u,v)^2 du dv, \tag{3}$$

subject to

$$\int_{\Omega_1 \cup \Omega_2} \lambda(u,v)^2 du dv = \int_{\Omega} du dv. \tag{4}$$

For high genus surfaces, if we want to assign zero points for a global conformal parameterization, different functionals should be formulated to minimize the conformal factor at the desired points. Suppose we want to assign $\{p_1, p_2, \cdots, p_n\} \subset S$ as zero points, where $U_i \subset \Omega$ is a neighborhood of p_i, and ω is a holomorphic 1-form. We define the functional as

$$E(\omega) = \sum_{i=1}^{n} \int_{U_i} \omega \wedge \bar{\omega}, \tag{5}$$

$\wedge$ represents the wedge product between holomorphic 1-forms. Intuitively, this functional measures the area of the neighborhoods of zero points on the parameter domain. If there is a holomorphic 1-form ω with zero points at all p_i's, then its $E(\omega)$ should be close to zero.

3.3 Optimal Holomorphic 1-form for High Genus Surface

A global conformal parameterization for a high genus surface can be obtained by integrating a holomorphic one form ω. Suppose $\{\omega_i, i = 1, 2, \cdots, 2g\}$ is a holomorphic 1-form basis, where an arbitrary holomorphic 1-form has the formula $\omega = \sum_{i=1}^{2g} \lambda_i \omega_i$. The energy for the parameterization is denoted $E(\omega)$, which is a function of the linear combination of coefficients λ_i. The necessary condition for the optimal holomorphic 1-form is straightforward, $\frac{\partial E}{\partial \lambda_i} = 0, i = 1, 2, \cdots, 2g$. If the Hessian matrix $(\frac{\partial^2 E}{\partial \lambda_i \partial \lambda_j})$ is positive definite, then E will reach the minimum. If the Hessian matrix is negative definite, E will be maximized. The traditional Newton's method can be applied for the optimization with the constraint that the total area in the parameter domain is fixed.

3.3.1 Uniform Global Conformal Parameterization

Given any holomorphic one-form ω, $\omega = \sum_{k=1}^{2g} \lambda_k \omega_k$, we require the total parameter area to be equal to the total area of the surface in R^3,

$$\sum_{[v_0,v_1,v_2] \in K_2} \frac{1}{2} |\omega([v_0, v_1]) \times \omega([v_1, v_2])| = \sum_{[v_0,v_1,v_2] \in K_2} S_{[v_0,v_1,v_2]}, \tag{6}$$

where $S_{[v_0,v_1,v_2]}$ is the area of face $[v_0, v_1, v_2]$ in R^3. The uniformity functional is defined as the sum of the squared area differences of faces,

$$E(\omega) = \sum_{[v_0,v_1,v_2] \in K_2} (\frac{1}{2} |\omega([v_0, v_1]) \times \omega([v_1, v_2])| - S_{[v_0,v_1,v_2]})^2. \tag{7}$$

Both the constraint and the energy functional are polynomials with respect to λ_i's. For example, the constraint can be reformulated as a quadratic form; if $c_{i,j} = \sum_{[v_0,v_1,v_2] \in K_2} \frac{1}{2} |\omega_i([v_0, v_1]) \times \omega_j([v_1, v_2])|$, then the constraint is $\sum_{i,j=1}^{2g} c_{ij} \lambda_i \lambda_j = const$.

We use Newton's method to optimize the energy with constraints. Because the energy is of degree 4, the extremal points are not unique. We randomly choose initial values for λ_i's, and choose the global optimal solution from local optimal ones. By minimizing the energy, we get the most uniform parameterization, for the purpose of comparison, we get the least uniform parameterization by maximizing the energy.

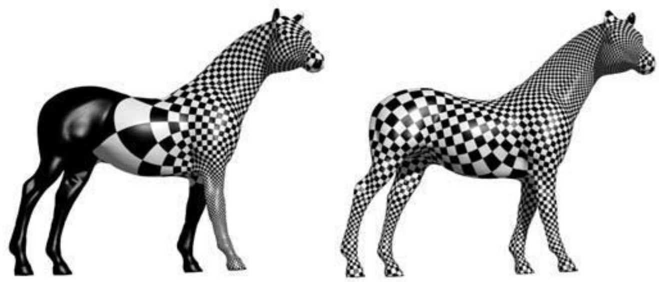
Figures 1 and 2 demonstrate the computation results. In figure 1, three cuts are introduced on the genus 0 bunny surface, two are on its ear tips, one is on the bottom, then the surface is double covered to become a genus 2 surface. In figure 2, the cuts are introduced at horse' feet and mouth, the double covered surface is of genus 4. The least uniform and the most uniform global parameterization are illustrated by using a checkerboard-texture map. Figure 5 uses the grid pattern to illustrate the computation results.

3.3.2 Emphasized Global Conformal Parameterization

Suppose we subdivide the whole surface into two regions D_0 and D_1. D_0 and D_1 themselves may be disconnected, with complicated topologies, and we want to maximize the parameter areas for D_0. Then, we define the *area energy* 3 as

$$E(\omega) = \frac{1}{2} \sum_{[v_0,v_1,v_2] \in D_0} |\omega([v_0, v_1]) \times \omega([v_1, v_2])| \tag{8}$$

with the same constraint in equation 6.



a. Least Uniform b. Most Uniform

Figure 2: Uniform Global Conformal Parameterization. Least uniform conformal parameterization, energy: 16.983e-5 (a). Most uniform conformal parameterization, energy: 7.878e-5(b).

The functional can be represented as a quadratic form directly. Let $c_{i,j} = \sum_{[v_0,v_1,v_2]\in D_0} |\omega_i([v_0,v_2]) \times \omega_j([v_1,v_2])|$, then the emphasized area energy is

$$E(\lambda_1,\lambda_2,\cdots,\lambda_{2g}) = \sum_{i,j=1}^{2g} c_{ij}\lambda_i\lambda_j. \quad (9)$$

By maximizing this functional, we get more samples on D_0 and less samples on D_1, and vice versa. The critical point is unique in general cases. We use Newton's method for the optimization with arbitrary initial values for the λ_i's.

Figure 1 demonstrates the optimization of the emphasized area energy for the bunny surface model. The surface is equally subdivided into the left part and the right part. Figure 1 (c) emphasizes the left part, and the parameter area of the left part is 83.48% of the total parameter area. Figure 1 (d) emphasizes the right part, the parameter area is 82.58% of the total parameter area.

3.4 Optimal Möbius Transform for Genus Zero Surface

For genus zero surfaces, there are no holomorphic one forms. We conformally map the surface to a unit sphere or a unit disk. Because the parameter domains are fixed, the constraint 6 is unnecessary.

We can still use the uniformity energy or the emphasized area energy, but the admissible transformations are changed to the Möbius transformations.

Topological sphere The Möbius transformation on the complex plane has the formula $\mu(z) = \frac{az+b}{cz+d}, ad-bc = 1, a,b,c,d \in \mathbb{C}$. A sphere can be conformally mapped to the complex plane by a stereographic projection $\tau : S^2 \to \mathbb{C}$, $\tau(x,y,z) = \frac{x}{1-z} + \sqrt{-1}\frac{y}{1-z}$.

A conformal automorphism ϕ of the sphere can be formulated as $\phi = \tau^{-1}\circ\mu\circ\tau$, We first compute a conformal map $\phi_0 : S \to S^2$ from the surface to the sphere, all admissible conformal mappings can be represented as $\phi_\mu = \tau^{-1}\circ\mu\circ\tau\circ\phi_0$.

The uniformity functional becomes

$$E(\mu) = \sum_{[v_0,v_1,v_2]\in K_2} (|\phi_\mu(v_0),\phi_\mu(v_1),\phi_\mu(v_2)| - S_{[v_0,v_1,v_2]})^2,$$

where $|p_0,p_1,p_2|$ represents the area of the triangle formed by p_0,p_1,p_2. This is a rational formula with respect to the coefficients of μ. We use Newton's method to optimize it without any constraints.

Similarly, the emphasized area energy is formulated by

$$E(\mu) = \sum_{[v_0,v_1,v_2]\in D_0} |\phi_\mu(v_0),\phi_\mu(v_1),\phi_\mu(v_2)|. \quad (10)$$

We use Newton's method to maximize the energy. Because the optimal solutions are not unique, we randomly choose the initial Möbius transformation μ_0, and use ϕ_{μ_0} as the initial parameterization.

Topological disk For the topological disk case, we use double covering to make it a symmetric topological sphere. However, we restrict the admissible transformations to be in a subgroup of the Möbius group, which preserves the symmetry; namely $\mu(\bar{z}) = \overline{\mu(z)}$.

The formula for such a Möbius transformation can be written as $\mu(z) = (az+b)/(\bar{b}z+\bar{a}), a\bar{a} - b\bar{b} = 1, a,b \in \mathbb{C}$.

Other steps are similar to those for the case of a topological sphere. Figure 3 illustrates a Möbius transformation from the disk to itself.

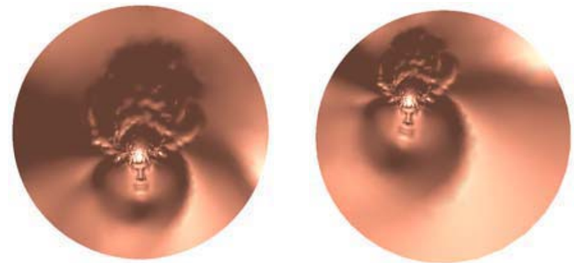


Figure 3: Möbius transformation from the unit disk to itself.

3.5 Topological Optimization

In this section, we introduce an automatic method to modify the topology of the surface to improve the uniformity of the parameterization.

For long tube shapes, such as fingers and tails, the area distortion is usually very big. We want to show that the problem cannot be solved by linear combination of the holomorphic one-form bases. We have to modify the conformal structure of the surface itself; namely, we either change the Riemannian metric or modify the topology.

First, we will demonstrate the fact that the conformal factor will increase exponentially on long tube shapes. Suppose we have a long thin cylinder and we conformally parameterize it. The center of the top is mapped to the origin. If we use polar coordinates (ρ,θ), then the conformal factor is a function dependent only on ρ because of symmetry. The Gaussian curvature K of the cylinder is zero, and

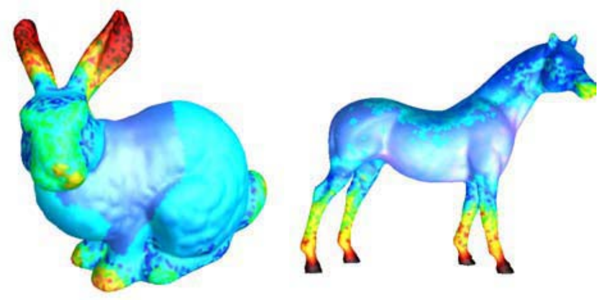
$$K(\rho,\theta) = \frac{1}{\lambda^2}\Delta\log\lambda = 0. \quad (11)$$

We can deduce $\lambda(\rho) = \exp(a\rho + b)$, where a,b are constants. No matter what kind of conformal parameterization we choose, the stretching is exponential. We have to change the topology of the surface by introducing a small boundary at the top of the cylinder, and then the conformal factor becomes constant.

Based on this observation, we design our greedy topological modification algorithm as follows. First we find the most uniform conformal parameterization for current surface. Second, we locate points with extremely high conformal factors. Third, we introduce a small slice at the neighborhoods of those points. Finally, its conformal structure is recomputed. We repeat the whole process until the uniformity energy is less than some threshold.

Estimating the Conformal Factor Suppose we have obtained a global conformal parameterization induced by a holomorphic one-form ω. The conformal factor for each vertex can be estimated by the following formula:

$$\lambda(v) = \frac{1}{n}\sum_{[v_i,v]\in K_1} \frac{|r(v_i) - r(v)|}{|\omega([v_i,v])|}, v_i, v \in K_0, \quad (12)$$



(a) (b)

Figure 4: Locating Extreme Point. Conformal factor is color encoded into bunny model (a). Conformal factor is color encoded into horse model (b). The extreme points of (a) are at the ear tips. The extreme points of (b) are at the feet.

where n is the valence of vertex v. In practice, since ω vanishes at degenerated points, we compute $\frac{1}{\lambda}$ instead of λ.

Locating the Extreme Points We locate the cluster of vertices with relatively high conformal factors, compute their center of gravity, and find the closest vertex to it. This vertex is an extreme point. Then we introduce a small slice through each extreme point, double cover the surface as described in [15], and compute a holomorphic 1-form basis. The optimal parameterization of the current topology is computed by minimizing the uniformity energy. We repeat the whole procedure until the energy is smaller than a specified threshold or converges to a limit.

Now we need to address the question of whether the uniformity would really be improved by this procedure. Suppose at step n, we get a surface S_n. Then any global conformal parameterization for S_n is also a global conformal parameterization for S_{n+1}, and the minimal uniformity energy of S_{n+1} denoted E_{n+1} is no greater than that of S_n. The sequence $\{E_0, E_1, E_{n+1}, \cdots\}$ is non-increasing and will converge to a limit. In practice, if the optimal uniformity energy does not decrease too much, the procedure will terminate.

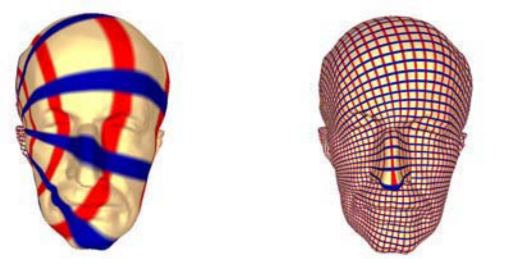
The results for topological optimization are illustrated in Figure 4. In Figure 4(a), the bunny is conformally mapped to a sphere. The conformal factors are color encoded where the red color means high conformal factor. The tips of ears are located accurately. A horse model is also processed. The feet, the mouth and the tip of ears are regions with high conformal factor. Then we introduce small boundaries to them and compute conformal structure for the modified surfaces.

Suppose a genus zero closed surface has k boundaries after topological optimization, and its double covering is of genus $k-1$. The parameterization can be further optimized by the method for high genus surfaces. Although we introduce more zero points, the quality of the parameterization is improved greatly. The boundary of small slices will be mapped to an iso-parametric line in the parameter domain; no singularities are introduced along the slices. In theory, the slices can be as small as possible to avoid affecting the rendering.

3.6 Zero Points Allocation

For a genus $g > 1$ surface, there are $2g-2$ zero points in a global conformal parameterization. In the neighborhood of zero points, the parameter areas of their neighborhoods are very small. If we want to construct geometry images from the surface, then these regions will be under-sampled. Then it is desirable to allocate zero points at some predetermined positions.

The positions of the zero points are globally related. They are determined by the conformal structure of the surface and are invariant under conformal mapping between surfaces. It is impossible to



(a). Least Uniform, energy: 4.231e-5

(b). Most Uniform, energy: 1.605e-5

Figure 5: Max Planck Head Model. Least and most uniform conformal parameterization.

assign $2g-2$ arbitrary points on the surface as the zero points.

Suppose ω is a holomorphic 1-form with $p_1, p_2, \cdots, p_{2g-2}$ as zero points; then $\omega(p_i) = 0, \forall i$. Let $\omega = \sum_{j=1}^{2g} \lambda_j \omega_j$; then we get the linear system

$$\sum_{j=1}^{2g} \lambda_j \omega_j(p_i) = 0, j = 1, 2, \cdots, 2g-2. \quad (13)$$

If $\{p_1, p_2, \cdots, p_{2g-2}\}$ is a set of zero points for some holomorphic one-form $\omega \neq 0$, it is necessary and sufficient that the matrix $(\omega_j(p_i))$ is degenerated.

In our discrete setting, $\omega = \sum_{i=1}^{2g} \lambda_i \omega_i$, and we use the following to approximate $|\omega(v)|, v \in K_0$.

$$\omega(v) = \frac{1}{n} \sum_{[v_i,v]\in K_1} \frac{|\omega([v_i,v])|}{|r(v_i) - r(v)|} = \frac{1}{n} \sum_{[v_i,v]\in K_1} \frac{|\sum_{j=1}^{2g} \lambda_j \omega_j([v_i,v])|}{|r(v_i) - r(v)|}.$$

Suppose we want to set n zero points $\{v_1, v_2, \cdots, v_n\}$, where $n < 2g-2$. Then we need to minimize the following energy

$$E(\omega) = \sum_{i=1}^{n} |\omega(v_i)|^2. \quad (14)$$

This functional is a quadratic form of $\lambda_1, \lambda_2, \cdots, \lambda_{2g}$ and can be solved easily using the conjugate gradient method. If n is not greater than g, then we can fix the zero points at the predetermined positions.

Figure 6 illustrates the two hole torus model. We predetermine the position of one zero point. By minimizing the energy in Equation 14, we can get the desired holomorphic one forms.

4 Results

The algorithms are developed using C++ on Windows XP platform, and tested with a dual processor PC with main frequency 2.8GHz. The statistics are illustrated in Table 1, where all meshes are after topological optimization. We also tested the algorithm stability by optimizing parameters for bunny meshes with different resolutions. The simplified meshes are generated using the progressive mesh method in [17]. The optimal parameterizations are consistent. The optimization uses Newton's method and stops when the energy difference between 2 consecutive iterations is less than a threshold.

We test our algorithms on several surface models acquired by laser scanning. The bunny model is of genus zero. The surface is sliced with 3 boundaries after topological optimization. The least uniform global conformal parameterization and most uniform global conformal parameterization results are illustrated in Figure 1 (a) and (b), respectively. Similarly, the horse model is of genus

Mesh	Vertices	Genus	Boundaries	Time (s)
eight	766	2	0	30
bunny	23996	0	3	150
horse	19994	0	7	250
Max-Planck	23609	0	1	180
Body	40000	0	5	350
David	200000	0	5	1800

Table 1: Performance for global conformal parameterization optimization.

zero and it has 5 boundaries after topological modification. The least uniform global conformal parameterization and the most uniform global conformal parameterization are illustrated in Figure 2 (a) and (b), respectively.

The Max Planck head surface in Figure 5 is a topological disk. Figure 5(a) illustrates the result with minimum uniformity energy and Figure 5 (b) illustrates the result with maximum uniformity energy.

The human body surface in Figure 7 has 5 boundaries. The double covering of this surface is of genus 4. We partition the whole surface to the left and right regions equally. The parameterization in Figure 7 (a) emphasizes the right region, which occupies 98.11% of the total parameter area. The parameterization in Figure 7 (b) emphasizes the left region, which occupies 96.1% of the total parameter area. The least uniform and the most uniform parameterization results are shown in Figure 7 (c) and (d) respectively.

Figure 6 illustrates the positions of zero points. We can get the desired holomorphic one-forms by minimizing Equation 14. The Michelangelo's David surface is illustrated in Figure 8. We control the zero points position using the method described in Section 3.6. In Figure 8(a), a zero point is located at the left upper arm near the shoulder. The same global conformal parameterization also has a zero point at his right upper arm near the shoulder as shown in (c). In Figure 8(b), there is a zero point under the left armpit. The same global conformal parameterization also gives a zero point at the right armpit, as shown in Figure 8(d).

5 Conclusion and Future Work

This work introduces systematic algorithms to optimize global conformal surface parameterizations. We define *uniformity energy* to measure the uniformity of the parameterization. We define *emphasized area energy* to measure the parameter area of regions of interest. We also define special functional to allocate zero points at the desired points. The problem of optimizing global conformal parameterizations is equivalent to searching for a desired Möbius transformation for genus zero surfaces and a desired holomorphic 1-form for high genus surfaces. We model global parameter optimizations as finite dimensional optimization problems, and use Newton's method to solve them. We also introduce algorithms to automatically modify the topology and allocate zero points at the specified positions to improve the quality of the global parameterization. The algorithms developed are efficient, intrinsic, practical, and versatile for different applications.

In the future, we will generalize the global conformal parameterizations to other parameterizations, such as Tuette, Stereo, Alexa parameterizations as in [11], and intrinsic parameterizations as in [6]. The generalization will be based on geometric differential equation theories.

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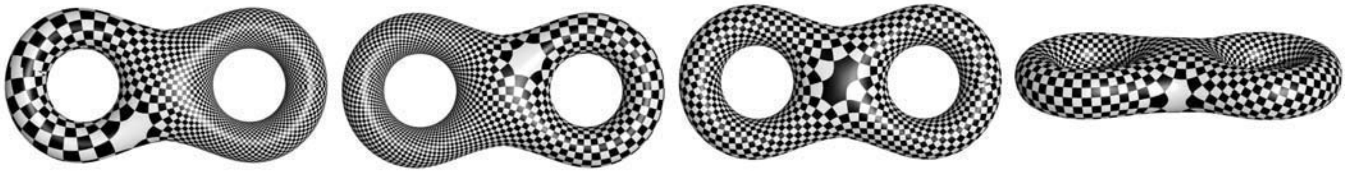
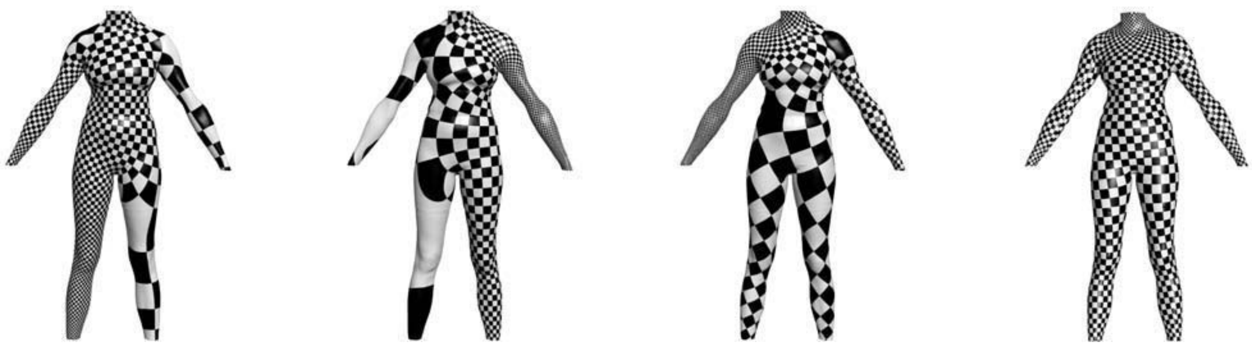
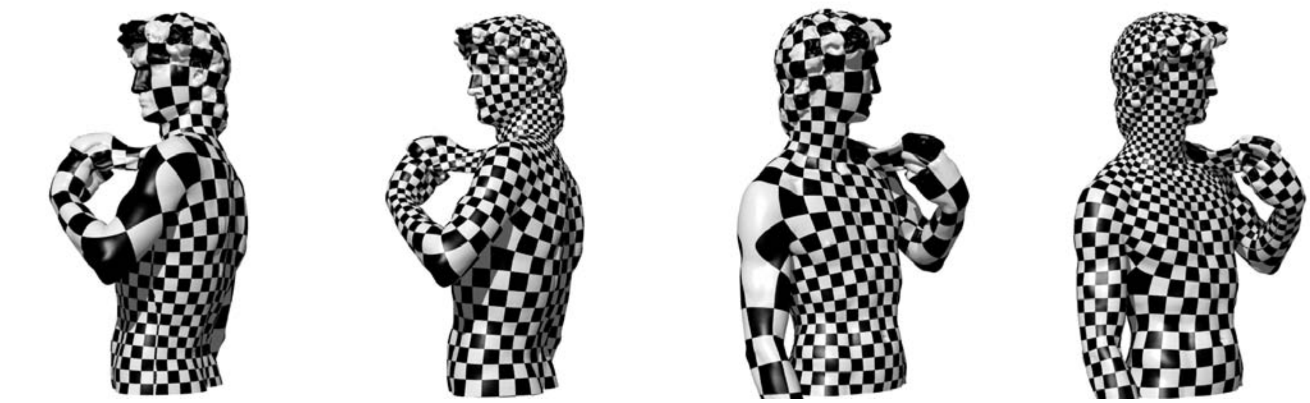


Figure 6: Two hole torus Model. Locate zero points at different positions



a. Emphasizing Left Part b. Emphasizing Right Part c. Least Uniform d. Most Uniform

Figure 7: Human body Model. (a) Maximizing the parameter areas of left, percentage: 98.11%. (b) Maximizing the parameter areas of right, percentage: 96.01%. (c)Least uniform conformal parameterization, energy: 2.798e-5(c). (d) Most uniform conformal parameterization, energy: 1.501e-5 (d).



a. Zero Point at Left Shoulder b. Zero Point at Left armpit c. Zero Point at Right shoulder d. Zero Point at Right armpit

Figure 8: Zero Point Allocation. Zero point is originally at left shoulder(a). Put zero point at Left armpit(b). Zero point is originally at right shoulder(c). Put zero point at right armpit(d).